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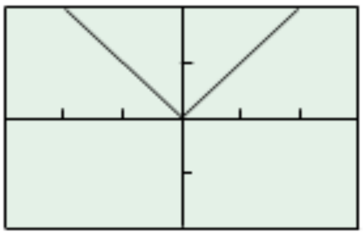
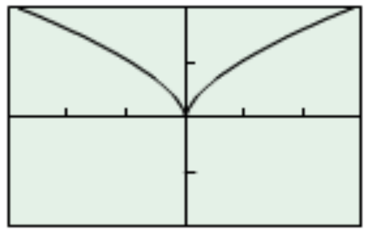
How $f'(a)$ Might Fail to Exist

1. a *corner,* where the one-sided derivatives differ; Example: $f(x) = |x|$



$[-3, 3]$ by $[-2, 2]$

Figure 3.11 There is a "corner" at $x = 0$.

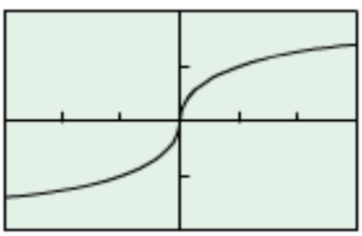


$[-3, 3]$ by $[-2, 2]$

Figure 3.12 There is a "cusp" at $x = 0$.

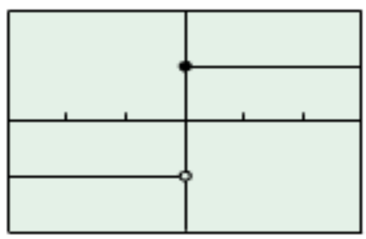
2. a *cusp,* where the slopes of the secant lines approach ∞ from one side and $-\infty$ from the other (an extreme case of a corner); Example: $f(x) = x^{2/3}$

3. a *vertical tangent,* where the slopes of the secant lines approach either ∞ or $-\infty$ from both sides (in this example, ∞); Example: $f(x) = \sqrt[3]{x}$



$[-3, 3]$ by $[-2, 2]$

Figure 3.13 There is a vertical tangent line at $x = 0$.



$[-3, 3]$ by $[-2, 2]$

Figure 3.14 There is a discontinuity at $x = 0$.

4. a *discontinuity* (which will cause one or both of the one-sided derivatives to be nonexistent). Example: The *Unit Step Function*

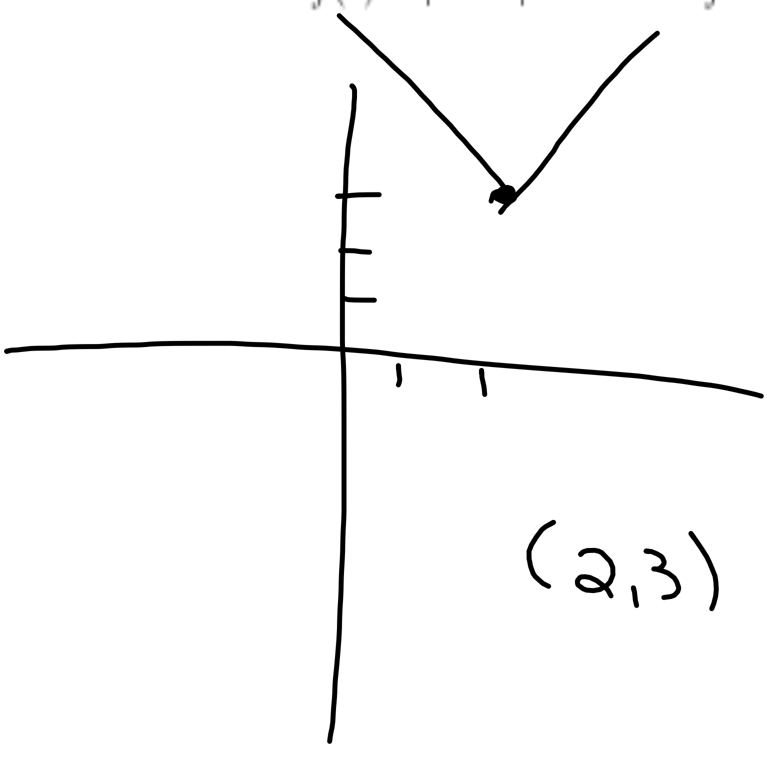
$$U(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

In this example, the left-hand derivative fails to exist:

$$\lim_{h \to 0^-} \frac{(-1) - (1)}{h} = \lim_{h \to 0^-} \frac{-2}{h} = \infty.$$

EXAMPLE 1 Finding Where a Function is not Differentiable

Find all points in the domain of $f(x) = |x - 2| + 3$ where f is not differentiable.



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EXPLORATION 1 Zooming in to "See" Differentiability

Is either of these functions differentiable at $x = 0$?

(a) $f(x) = |x| + 1$ **(b)** $g(x) = \sqrt{x^2 + 0.0001} + 0.99$

1. We already know that f is not differentiable at $x = 0$; its graph has a corner there. Graph f and zoom in at the point $(0, 1)$ several times. Does the corner show signs of straightening out?

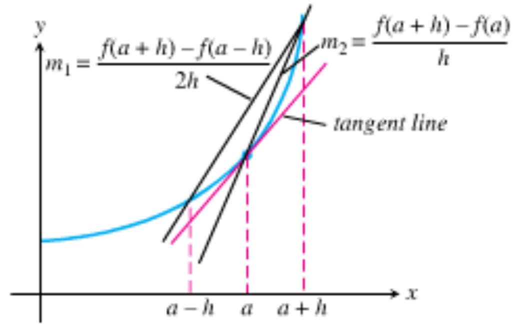
2. Now do the same thing with g. Does the graph of g show signs of straightening out? We will learn a quick way to differentiate g in Section 3.6, but for now suffice it to say that it *is* differentiable at $x = 0$, and in fact has a horizontal tangent there.

3. How many zooms does it take before the graph of g looks exactly like a horizontal line?

4. Now graph f and g *together* in a standard square viewing window. They appear to be identical until you start zooming in. The differentiable function eventually straightens out, while the nondifferentiable function remains impressively unchanged.

is often a good numerical approximation of $f'(a)$. However, as suggested by Figure 3.16, the same value of h will usually yield a *better* approximation if we use the **symmetric difference quotient**

$$\frac{f(a+h)-f(a-h)}{2h},$$



$$\text{NDER } f(a) = \frac{f(a+0.001)-f(a-0.001)}{0.002}.$$

$x^3, x, 2$

EXAMPLE 2 Computing a Numerical Derivative

Compute NDER $(x^3, 2)$, the numerical derivative of x^3 at $x = 2$.

Put x^3 in Y_1. Then quit to a clear screen.

Push MATH 8 (nDeriv)

$$\frac{d}{dx}(f(x))\Big|_{x=a}$$

Type Y_1 where you see f(x) and 2 where you see a.

Note that both results are the same and equal to that shown in Example 2. Both Mathprint and Classic modes give the *same numerical approximation* because they are identical computationally on the calculator! Mathprint mode is *not* a Computer Algebra System (CAS) mode giving exact results like $\frac{d}{dx}x^3 = 3x^2$ or 12 when $x = 2$. Also remember that in our textbook we use **NDER (*f*(*x*), *a*)** to represent the numerical derivative of $f(x)$ at $x = a$, as defined on this page.

EXAMPLE 3 Fooling the Symmetric Difference Quotient

Compute NDER $(|x|, 0)$, the numerical derivative of $|x|$ at $x = 0$.

Calculator is wrong!

EXPLORATION 2 Looking at the Symmetric Difference Quotient Analytically

Let $f(x) = x^2$ and let $h = 0.01$.

1. Find

$$\frac{f(10 + h) - f(10)}{h}.$$

   How close is it to $f'(10)$?

2. Find

$$\frac{f(10 + h) - f(10 - h)}{2h}.$$

   How close is it to $f'(10)$?

3. Repeat this comparison for $f(x) = x^3$.

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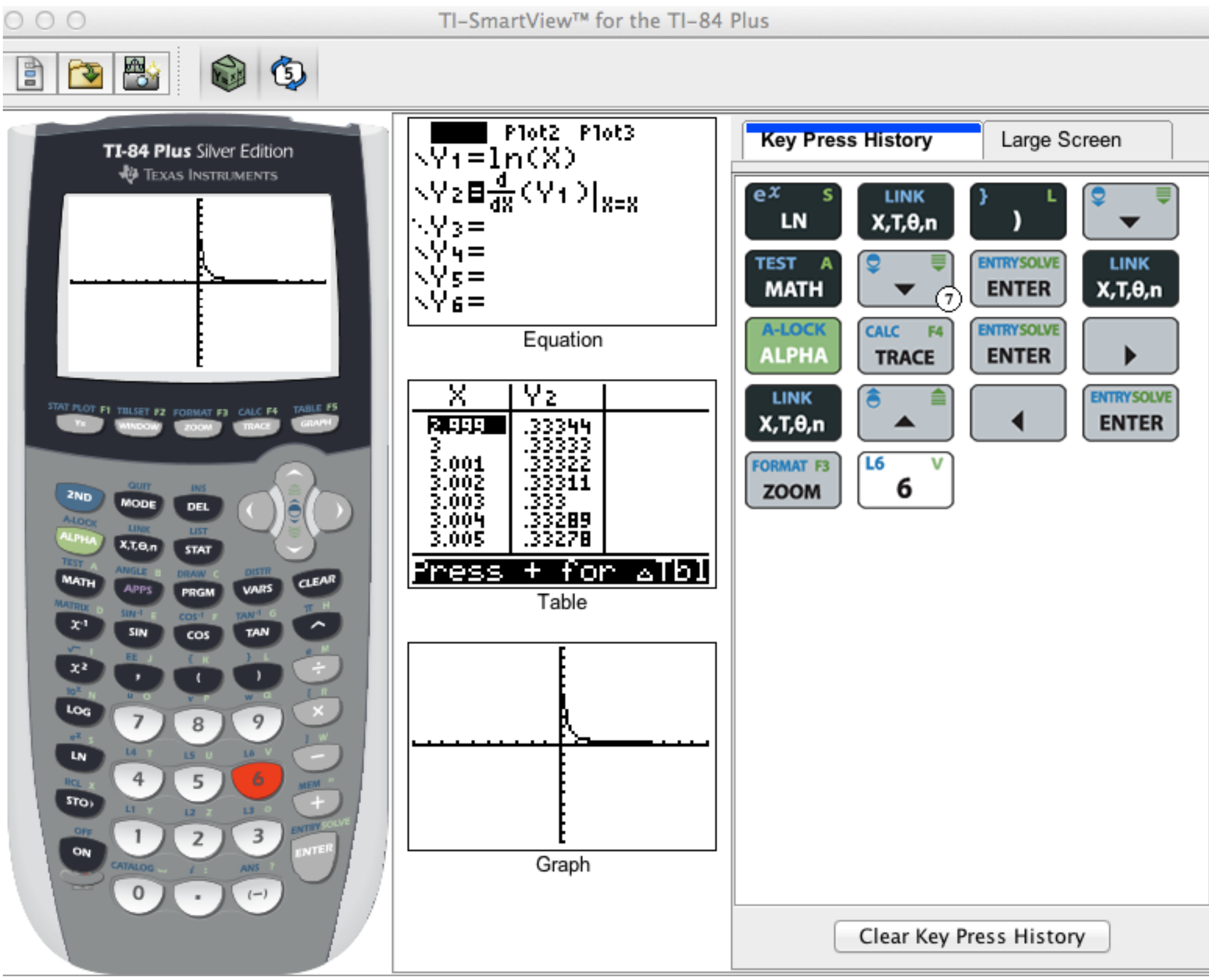
EXAMPLE 4 Graphing a Derivative Using NDER

Let $f(x) = \ln x$. Use NDER to graph $y = f'(x)$. Can you guess what function $f'(x)$ is by analyzing its graph?



THEOREM 1 Differentiability Implies Continuity

If f has a derivative at $x = a$, then f is continuous at $x = a$.

The converse of Theorem 1 is false, as we have already seen. A continuous function might have a corner, a cusp, or a vertical tangent line, and hence not be differentiable at a given point.

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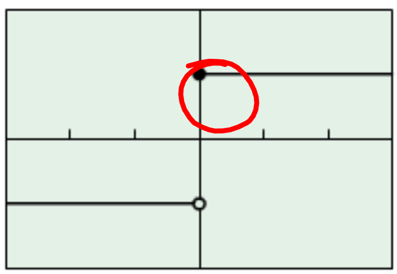
THEOREM 2 Intermediate Value Theorem for Derivatives

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

EXAMPLE 5 Applying Theorem 2

Does any function have the Unit Step Function (see Figure 3.14) as its derivative?

no



[−3, 3] by [−2, 2]

the derivative exists everywhere

but does not = values between

−1 and 1